

Getting at the Quark Mass Matrices

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Abstract

We present a class of *Ansätze* for the up and down quark mass matrices which leads approximately to: $|V_{us}| \sim \sqrt{m_d/m_s}$, $|V_{cb}| \sim m_s/m_b$, and $|V_{ub}/V_{cb}| \sim \sqrt{m_u/m_c}$. Sizes of the Kobayashi-Maskawa matrix elements are controlled solely by quark mass ratios. In particular, we introduce no other small parameter and our results do not rely on delicate cancellation.

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1 Introduction

Within the context of the standard electroweak model, the origin of quark masses and the Kobayashi-Maskawa (KM) flavor mixing matrix [1] remains a mystery . Therefore, a search for the origin of the KM matrix and quark masses may show us how to get at the physics beyond the standard model.

There are 36 parameters in 3×3 up and down quark mass matrices. In contrast, there are 10 experimental constraints - 6 quark masses and 4 KM matrix elements. As there are too many parameters, it is often more constructive to start with a theory - predict the mass matrices and check them with experiments. In this paper, we take a different approach and present a class of *Ansätze* for the mass matrices.

Current experimental data suggest the following relations between quark masses and the KM matrix elements [2]:

$$\begin{aligned} |V_{us}| &\approx \sqrt{\frac{m_d}{m_s}} \approx 0.22 , \\ |V_{cb}| &\approx \frac{m_s}{m_b} \approx 0.02 \sim 0.06 , \\ \left| \frac{V_{ub}}{V_{cb}} \right| &\approx \sqrt{\frac{m_u}{m_c}} \approx 0.035 \sim 0.09 . \end{aligned} \tag{1}$$

These relations may be an accident. We take a view that the Nature is giving us a clue. We thus take them seriously, and give a wide class of mass matrices which can lead to relationships of this form.

As mentioned above, 26 out of 36 parameters of quark mass matrices have nothing to do with quark masses or KM matrix elements. It is convenient to remove all but essential parameters out of the mass matrices. As first shown in Ref.[3], it is always possible to find a weak basis where arbitrary 3×3 quark mass matrices take the nearest-neighbor interaction (NNI) form:

$$M = m_3 M' = m_3 \begin{pmatrix} 0 & a & 0 \\ c & 0 & b \\ 0 & d & e \end{pmatrix} . \tag{2}$$

Here we have rescaled all matrix elements of M by use of the 3rd generation mass, m_3 . By redefining the phases of left-handed and right-handed quark fields, one can choose all elements

of M' to be real and non-negative. In this case, a diagonal phase matrix

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \exp(i\theta_2) & 0 \\ 0 & 0 & \exp(i\theta_3) \end{pmatrix} \quad (3)$$

will enter the flavor mixing matrix. Diagonalizing MM^T by the orthogonal transformation

$$O^T MM^T O = \begin{pmatrix} m_1^2 & 0 & 0 \\ 0 & m_2^2 & 0 \\ 0 & 0 & m_3^2 \end{pmatrix}, \quad (4)$$

we can obtain the KM matrix V for flavor mixings:

$$V = O_u^T P O_d, \quad (5)$$

where the subscripts “u” and “d” stand for up and down, respectively.

In this analysis, V contains 12 free parameters: 6 of them are determined from quark mass eigenvalues, and 4 from the experimental data on V . Thus all but two of the parameters in M_u and M_d can be determined.

Now, let us require that M_u and M_d given in Eq.(2) reproduce correct quark mass eigenvalues. For each quark sector we introduce four real (positive) parameters p , q , y and z , which satisfy the relations

$$p = \frac{m_1^2 + m_2^2}{m_3^2}, \quad q^2 = \frac{m_1 m_2}{m_3^2}; \quad (6a)$$

and

$$a = \frac{qz}{y}, \quad c = \frac{q}{yz}, \quad e = y^2. \quad (6b)$$

As a result, the mass matrix elements b and d can be given by

$$\begin{aligned} b &= \sqrt{\frac{p+1-y^4 \pm R}{2}} - \frac{q^2}{y^2 z^2}, \\ d &= \sqrt{\frac{p+1-y^4 \mp R}{2}} - \frac{q^2 z^2}{y^2}, \end{aligned} \quad (7)$$

where

$$R \equiv \sqrt{(1+p-y^4)^2 - 4(p+q^4) + 4q^2 y^2 \left(z^2 + \frac{1}{z^2}\right)}. \quad (8)$$

The quark mass matrix M' turns out to be the following form [4]:

$$M' = \begin{pmatrix} 0 & \frac{qz}{y} & 0 \\ \frac{q}{yz} & 0 & b \\ 0 & d & y^2 \end{pmatrix}, \quad (9)$$

whose structure is characterized by the dimensionless parameters y and z . Note that b and d have two different solutions as given in Eq.(7). Subsequently, we refer the case that b takes plus (minus) sign and d takes minus (plus) sign to case (I) (case (II)).

2 A class of *Ansätze*

Our *Ansätze* are to take the NNI form for the up and down quark matrices, and require that their parameters satisfy the following conditions:

$$y \sim O(1), \quad z \sim O(1), \quad (1 - y^4)^2 \gg 2p(1 + y^4). \quad (10)$$

The form given in Eq.(9) is completely general. The *Ansätze* made in Eq.(10) imply that when quark mass matrices are transformed into the NNI form, the parameters y and z are restricted to a certain region.

It may be that this class of mass matrices have nothing to do with the Nature. For that matter, singling out the NNI basis on which we assume Eq.(10) may lead us astray. This assumption must be judged by its consequences.

3 Results

In this approximation, we get

$$R \approx (1 - y^4) - \frac{1 + y^4}{1 - y^4} p. \quad (11)$$

Then we obtain the magnitudes of b and d as follows:

$$\text{case (I) : } b \approx \sqrt{1 - y^4}, \quad d \approx \sqrt{\frac{p}{1 - y^4}}; \quad (12a)$$

$$\text{case (II) : } b \approx \sqrt{\frac{p}{1 - y^4}}, \quad d \approx \sqrt{1 - y^4}. \quad (12b)$$

We insist that either case (I) or case (II) is chosen for both M_u and M_d . This leads to similar structure for both M_u and M_d . Also, by doing so, the diagonal elements of the KM matrix are neary unity. Note that $b \gg d$ in case (I) is not favored by current data. To see this point more clearly, we compute V_{cb} in case (I) and obtain

$$V_{cb} \approx y_u^2 \sqrt{1 - y_d^4} \exp(i\theta_2) - y_d^2 \sqrt{1 - y_u^4} \exp(i\theta_3) \quad (13)$$

at the lowest-order expansion in p and q . We find that this result is quite unsatisfactory. It is nearly independent of the quark mass ratios; and it has to involve large cancellation between the $\exp(i\theta_2)$ and $\exp(i\theta_3)$ terms to agree with the measured value of $|V_{cb}|$. We, therefore focus our attention on case(II).

By use of Eq.(11) and Eq.(12b), we calculate the KM matrix V by keeping only the leading terms in p and q . The diagonal elements of V can be given as

$$V_{ud} \approx 1, \quad V_{cs} \approx \exp(i\theta_2), \quad V_{tb} \approx \exp(i\theta_3) \quad (14)$$

in lowest-order approximations. We also find

$$\begin{aligned} V_{us} &\approx -y_d z_d \sqrt{\frac{m_d}{m_s}} + y_u z_u \sqrt{\frac{m_u}{m_c}} \exp(i\theta_2), \\ V_{cd} &\approx -y_u z_u \sqrt{\frac{m_u}{m_c}} + y_d z_d \sqrt{\frac{m_d}{m_s}} \exp(i\theta_2); \\ V_{cb} &\approx \frac{y_d^2}{\sqrt{1-y_d^4}} \frac{m_s}{m_b} \exp(i\theta_2) - \frac{y_u^2}{\sqrt{1-y_u^4}} \frac{m_c}{m_t} \exp(i\theta_3), \\ V_{ts} &\approx \frac{y_u^2}{\sqrt{1-y_u^4}} \frac{m_c}{m_t} \exp(i\theta_2) - \frac{y_d^2}{\sqrt{1-y_d^4}} \frac{m_s}{m_b} \exp(i\theta_3); \\ V_{ub} &\approx \frac{z_d \sqrt{1-y_d^4}}{y_d} \sqrt{\frac{m_d}{m_s}} \frac{m_s}{m_b} + y_u z_u \sqrt{\frac{m_u}{m_c}} \left[\frac{y_d^2}{\sqrt{1-y_d^4}} \frac{m_s}{m_b} \exp(i\theta_2) - \frac{1}{y_u^2 \sqrt{1-y_u^4}} \frac{m_c}{m_t} \exp(i\theta_3) \right], \\ V_{td} &\approx \frac{z_u \sqrt{1-y_u^4}}{y_u} \sqrt{\frac{m_u}{m_c}} \frac{m_c}{m_t} + y_d z_d \sqrt{\frac{m_d}{m_s}} \left[\frac{y_u^2}{\sqrt{1-y_u^4}} \frac{m_c}{m_t} \exp(i\theta_2) - \frac{1}{y_d^2 \sqrt{1-y_d^4}} \frac{m_s}{m_b} \exp(i\theta_3) \right]. \end{aligned} \quad (15)$$

To transform the KM matrix V obtained above to a more familiar Wolfenstein form [5], we make the following phase rotations:

$$\begin{aligned} c &\rightarrow c \exp(i\theta_3), & t &\rightarrow t \exp[i(\theta_3 - \varphi)], \\ u &\rightarrow u \exp(i\phi), & d &\rightarrow d \exp(i\phi), \\ b &\rightarrow b \exp(-i\varphi), \end{aligned} \quad (16)$$

where the phase shifts ϕ and φ are defined as

$$\begin{aligned} \phi &= \arctan \left[\frac{\sin \theta_2}{\cos \theta_2 - \frac{y_d z_d}{y_u z_u} \sqrt{\frac{m_c m_d}{m_u m_s}}} \right], \\ \varphi &= \arctan \left[\frac{\sin(\theta_3 - \theta_2)}{\cos(\theta_3 - \theta_2) - \frac{y_d^2 \sqrt{1-y_u^4} m_t m_s}{y_u^2 \sqrt{1-y_d^4} m_c m_b}} \right]. \end{aligned} \quad (17)$$

Accordingly, we get

$$V \approx \begin{pmatrix} 1 - \frac{1}{2}\lambda^2 & \lambda & V_{ub} \\ -\lambda & 1 - \frac{1}{2}\lambda^2 & A\lambda^2 \\ V_{td} & -A\lambda^2 & 1 \end{pmatrix}, \quad (18)$$

where

$$\begin{aligned} \lambda &= \sqrt{y_u^2 z_u^2 \frac{m_u}{m_c} + y_d^2 z_d^2 \frac{m_d}{m_s} - 2y_u z_u y_d z_d \sqrt{\frac{m_u m_d}{m_c m_s}} \cos \theta_2}, \\ A\lambda^2 &= \sqrt{\frac{y_u^4}{1 - y_u^4} \frac{m_c^2}{m_t^2} + \frac{y_d^4}{1 - y_d^4} \frac{m_s^2}{m_b^2} - \frac{2y_u^2 y_d^2}{\sqrt{1 - y_u^4} \sqrt{1 - y_d^4}} \frac{m_c m_s}{m_t m_b} \cos(\theta_3 - \theta_2)}; \end{aligned} \quad (19)$$

and

$$\begin{aligned} V_{ub} &\approx y_u z_u \sqrt{\frac{m_u}{m_c}} \left[\frac{y_d^2}{\sqrt{1 - y_d^4}} \frac{m_s}{m_b} - \frac{1}{y_u^2 \sqrt{1 - y_u^4}} \frac{m_c}{m_t} \exp[i(\theta_3 - \theta_2)] \right] \exp[i(\theta_2 - \phi - \varphi)] \\ &\quad + \frac{z_d \sqrt{1 - y_d^4}}{y_d} \sqrt{\frac{m_d}{m_s} \frac{m_s}{m_b}} \exp[-i(\phi + \varphi)], \\ V_{td} &\approx y_d z_d \sqrt{\frac{m_d}{m_s}} \left[\frac{y_u^2}{\sqrt{1 - y_u^4}} \frac{m_c}{m_t} \exp[i(\theta_2 - \theta_3)] - \frac{1}{y_d^2 \sqrt{1 - y_d^4}} \frac{m_s}{m_b} \right] \exp[i(\phi + \varphi)] \\ &\quad + \frac{z_u \sqrt{1 - y_u^4}}{y_u} \sqrt{\frac{m_u}{m_c} \frac{m_c}{m_t}} \exp[i(\phi + \varphi - \theta_3)]. \end{aligned} \quad (20)$$

4 Discussion

Now let us compare these results with Eq.(1). For simplicity, we shall drop the terms which are suppressed by $1/m_t$. We then get

$$A\lambda^2 \approx \frac{y_d^2}{\sqrt{1 - y_d^4}} \frac{m_s}{m_b}, \quad (21)$$

$$\left| \frac{V_{td}}{V_{ts}} \right| \approx \frac{z_d}{y_d^3} \sqrt{\frac{m_d}{m_s}}, \quad (22)$$

$$\left| \frac{V_{ub}}{V_{cb}} \right| \approx \sqrt{y_u^2 z_u^2 \frac{m_u}{m_c} + \frac{z_d^2 (1 - y_d^2)^2}{y_d^6} \frac{m_d}{m_s} + \frac{2y_u z_u z_d (1 - y_d^4)}{y_d^3} \sqrt{\frac{m_u m_d}{m_c m_s}} \cos \theta_2}. \quad (23)$$

Considering the fact that Eq.(1) is a simple guess, we find that these results together with the expression for $V_{us} = \lambda$ given in Eq.(19), are quite consistent with Eq.(1).

For the purpose of illustration, let us estimate the magnitudes of the relevant parameters. There are six unknown parameters (y_u , y_d , z_u , z_d , θ_2 and θ_3), and only four of them can be determined from the experimental data on V . In the limit that top quark is very heavy, we see that θ_3 and y_u/z_u are the two parameters which remain undetermined.

(i) With the help of the experimental value $|V_{cb}| \approx 0.04$ [2] and the scale-independent result $m_b/m_s \approx 34$ [6], Eq.(21) leads to $y_d \approx 0.9$.

(ii) Current data together with the unitarity of V suggests $0.13 \leq |V_{td}/V_{ts}| \leq 0.35$ [7], while the magnitude of m_s/m_d is allowed to be in the range 17 to 25 [2]. Typically taking $m_s/m_d \approx 21$, $y_d \approx 0.9$ and $|V_{td}/V_{ts}| \approx 0.25$, we obtain $z_d \approx 0.84$ from Eq.(22).

(iii) Taking $|V_{us}| \approx 0.22$, $|V_{ub}/V_{cb}| \approx 0.08$ [2] and $m_u/m_c \approx 0.004$ [8], we find $y_u z_u \approx 1.3$ and $\theta_2 \approx 123^\circ$ from Eq.(23). Accordingly, one obtains $\phi \approx 162^\circ$ from Eq.(17).

We stress that

$$y_d \approx 0.9, \quad z_d \approx 0.84, \quad y_u z_u \approx 1.3 \quad (24)$$

are in agreement with assumptions given in Eq.(10).

The Wolfenstein parameters can be written as follows. First note that

$$\left| \frac{V_{ub}}{V_{cb}} \right| \approx \left| \lambda \exp(i\phi) + \frac{z_d}{y_d^3} \sqrt{\frac{m_d}{m_s}} \right|, \quad (25)$$

which is consistent with unitarity of V . From Eq.(25) we deduce

$$\rho \approx 1 + \frac{z_d}{y_d^3 \lambda} \sqrt{\frac{m_d}{m_s}} \cos \phi, \quad \eta \approx \frac{z_d}{y_d^3 \lambda} \sqrt{\frac{m_d}{m_s}} \sin \phi. \quad (26)$$

In the $\rho - \eta$ plane, the unitarity triangle formed by three sides $V_{ub}^* V_{ud}$, $V_{cb}^* V_{cd}$ and $V_{tb}^* V_{td}$ can be rescaled into a simpler one with vertices $A(\rho, \eta)$, $B(1, 0)$ and $C(0, 0)$ [2]. Its inner angles $\phi_1 \equiv \angle ABC$ and $\phi_2 \equiv \angle BAC$ are approximately given as

$$\phi_1 \approx 180^\circ - \phi, \quad \phi_2 \approx \arctan \left[\frac{-\sin \theta_2}{\cos \theta_2 + \frac{(1 - y_d^4) z_d}{y_u z_u y_d^3} \sqrt{\frac{m_c m_d}{m_u m_s}}} \right]. \quad (27)$$

Similarly, one can calculate the rephasing-invariant measure of CP violation in the 3×3 quark mixing matrix (the so-called Jarlskog parameter [9]). By use of Eq.(15) or Eq.(18), we find

$$J = \text{Im} (V_{us} V_{cb} V_{ub}^* V_{cs}^*) \approx \frac{y_u y_d z_u z_d}{1 - y_d^4} \sqrt{\frac{m_u}{m_c}} \sqrt{\frac{m_d}{m_s}} \left(\frac{m_s}{m_b} \right)^2 \sin \theta_2. \quad (28)$$

With the help of the rough results obtained above, we get $J \approx 2.9 \times 10^{-5}$, consistent with the present experimental expectation.

Note that the result of V_{cb} obtained in Eq.(15) or Eq.(21) is quite different from that predicted by the Fritzsch *Ansatz* (one of the simplest patterns of the NNI mass matrices [10]):

$$|V_{cb}|_{\text{Fr}} \approx \left| \sqrt{\frac{m_s}{m_b}} - \sqrt{\frac{m_c}{m_t}} \exp(i\theta_3) \right|, \quad (29)$$

which does not agree with the experimental values of $|V_{cb}|$ and the top quark mass m_t . This difference is easily understood, since Eq.(29) arises from $b = d \approx \sqrt{m_2/m_3}$ while Eq.(21) is guaranteed by the condition in Eq.(12b) with $b \propto m_2/m_3$. Nevertheless, we find that $z = 1$, assumed in the Fritzsch mass matrices, remains to be an interesting choice. It leads to a consistent set of values for V_{us} , V_{cd} , V_{ub} and V_{td} in Eq.(15). One can conclude that in the NNI basis $d \gg b$, i.e., a large asymmetry between (2,3) and (3,2) elements of M' , is the necessary condition for M' to accommodate the experimental value of V_{cb} . This point has also been noticed in Refs.[11] and [12], where $d_u \gg b_u$, $d_d = b_d$ and $d_u = e_u \gg b_u$, $d_d = e_d \gg b_d$ are respectively taken. Finally it is worth mentioning that Ito and Tanimoto [13] have studied another parameter space of the generic NNI mass matrices and their analytical results for V_{cb} , V_{ts} , V_{ub} and V_{td} are different from ours.

Without loss of generality, it is more common to construct Hermitian mass matrices from a theory. If we transform M' into another weak basis in which the resultant quark mass matrix takes the Hermitian form, then its (2,2) element should be nonvanishing [14], as required by the inequality of b and d in M' .

5 On the scale dependence

Note that the mass matrix M' depends upon the underlying renormalization scale μ . In our discussion above, we took M' defined at a low-energy scale μ_0 ($\leq m_Z$). The running of M' from μ_0 to an arbitrary renormalization point μ (e.g., the grand unification theory scale) depends on a specific model of spontaneous symmetry breaking. In general, the texture zeros of M' , as well as the parallel structures of M'_u and M'_d , may disappear after the evolution [15].

Of course, the KM matrix elements are scale dependent too. A detailed analysis has shown that $|V_{ub}|$, $|V_{cb}|$, $|V_{td}|$ and $|V_{ts}|$ have identical evolution from μ_0 to μ ($> \mu_0$), and the running effects of m_u/m_c , m_d/m_s , $|V_{us}|$ and $|V_{cd}|$ are negligibly small [16]. Therefore, the approximate

results for $|V_{ub}/V_{cb}|$ in Eq.(22) and $|V_{td}/V_{ts}|$ in Eq.(23) are independent of the renormalization scale μ . Also, $|V_{cb}|$ and m_s/m_b may have the same running behavior (dominated by the top quark Yukawa coupling) in the standard electroweak model or its supersymmetric extension with small $\tan\beta$ (the ratio of Higgs vacuum expectation values) [16]. In this case, the approximate formula for $|V_{cb}|$ in Eq.(21) is scale independent. For large $\tan\beta$ (e.g., $\tan\beta \geq m_t/m_b$), however, the terms proportional to m_c/m_t in Eq.(15) may be non-negligible and our approximate results in Eq.(21), Eq.(22) and Eq.(23) would involve larger errors. At least, an additional evolution factor has to be introduced into Eq.(21) due to the significantly different renormalization effects between $|V_{cb}|$ and m_s/m_b . One can see that the scale dependence of J in Eq.(28) is dominantly controlled by that of m_s/m_b .

6 Summary

Noticing that the KM matrix elements are numerically equal to certain functions of quark mass ratios, we have presented a class of *Ansätze* for the quark mass matrices. It is given in the NNI form where all irrelevant parameters of the mass matrices have been removed. The main assumption is that all hierarchy present in the KM matrix elements are due to quark mass ratios and there is no other small parameter or delicate cancellation.

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